MOTION OF A PARTICLE IN A VORTEX CHAMBER

M. A. Gol'dshtik and V. N. Sorokin

Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, Vol. 9, No. 6, pp. 149–152, 1968

We consider a vortex-sink gas flow in a space between two parallel plates perpendicular to the rotation axis and separated by a distance h. We assume a spherical particle of diameter d is added to the flow. If the particle does not interact with the plane, then as it moves along with the flow under steadystate conditions it will trace out a circular orbit whose radius is found from the equation for the balance of the radial forces acting on the particle:

$$\frac{1}{6}\pi d^3\rho_1\frac{v_\varphi^2}{r}=\xi\,\frac{\pi d^2}{4}\frac{\rho v_r^2}{2}. \qquad (1)$$

Equation (1) reflects the fact that the centrifugal force is balanced by the radial drag. Here v_r and v_φ are the radial and tangential components of the flow velocity, ρ and ρ_1 are the gas and particle densities, and ξ is the particle drag coefficient. For a vortex sink of an incompressible liquid, we have

$$v_\varphi=\frac{\Gamma}{2\pi r},\quad v_r=-\frac{Q}{2\pi r}.$$

Using this and Eq. (1), we find the equilibrium orbit to have a radius

$$r=\frac{4}{3\xi}\frac{\rho_1}{\rho}\left(\frac{\Gamma}{Q}\right)^2 d. \qquad (2)$$

Here Γ and Q are the circulation and flow per unit length of the vortex sink.

An experimental study of the motion of a particle in an vortex chamber does not verify Eq. (2); two characteristic properties of this motion are found. First, under steady-state conditions, the particle does not rotate along with the flow, but lags behind it significantly; second, the particle interacts intensely with the chamber walls, undergoing many collisions of a more or less periodic nature with both walls. These two features are obviously related to each other; i.e., the lagging of the particle behind the flow is due to loss of angular momentum by the particle during its collision with the wall. The condition for steady-state motion would be that this loss be balanced by the twisting effect of the flow. At a sufficiently high flow rate the particle velocity does not depend on the orientation of the chamber in the gravitational field; therefore, we neglect this orientation below.

As a result of a collision of the particle with the wall, the particle will not only lose normal and tangential components of its initial momentum, but it will also acquire an angular velocity ω; accordingly, a transverse Magnus force must be added to the forces acting on the particle. The magnitude of this Magnus force can be approximated on the basis of the following arguments.

Consider a sphere of radius a rotating at angular velocity ω is encompassed by a flow at velocity V directed perpendicular to the rotation axis. At a distance z from the center of the sphere, we single out a layer of thickness dz perpendicular to the rotation axis. By analogy with airfoil theory for an airfoil of finite wing span, we assume the "planar-cross section" hypothesis; i.e., we assume that the flow pattern near a given layer does not differ from the planar flow of a circle with circulation $\Gamma = 2\pi\omega(a^2 - z^2)$. The transverse force dF acting on the layer dr is calculated from the Zhukovskii equation $dF = \rho v\Gamma dz$ ($\rho v\Gamma$ is the force per unit length of an infinite cylinder). For the total force F, we find

$$F=2\rho v\int_0^a\Gamma\,dz=2\,\frac{4}{3}\pi a^3\rho\omega v=2m\,\frac{\rho}{\rho_1}\omega v,$$

where m is the mass of the particle.

The secondary flow from the equator toward the poles which arises around the rotating sphere [1] has not been taken into account, so this equation should yield slightly exaggerated values of the transverse x force. However, an experimental check of this equation [2] shows that the correction may be neglected for rapidly rotating small particles. Taking into account the direction of the Magnus force, we finally write

$$F = 2m\frac{\rho}{\rho_1}[\boldsymbol{\omega} \times \mathbf{u}] \qquad (\mathbf{u} = \mathbf{w} - \mathbf{v}). \tag{3}$$

Here **u** and **w** are the particle's relative and absolute velocities, respectively. When the transverse force is taken into account, the equation of motion of the particle, which is experiencing quadratic drag, becomes

$$\frac{d\mathbf{w}}{dt} = 2\frac{\rho}{\rho_1}[\boldsymbol{\omega}, \mathbf{u}] - \frac{3}{4}\frac{\xi}{d}\frac{\rho}{\rho_1}|u|\mathbf{u}, \qquad |\mathbf{u}| = \sqrt{(w_r - v_r)^2 + (w_\varphi - v_\varphi)^2 + (w_z - v_z)^2}, \tag{4}$$

where $w_r = v_z = 0$. We assume $|u|$ is constant. When the particle lags only slightly behind the flow, we have $|u| \approx |v_r|$; when $w_\varphi \ll v_\varphi$, we have

$$|u| = \sqrt{v_r^2 + v_\varphi^2}.$$

We now imagine that we unfold the cylindrical surface in which the particle is moving into a plane; i.e., we consider the planar motion of the particle between two parallel planes under the influence of a translational flow at velocity $v = v_\varphi$. The familiar justification for this approximation is the fact that a particle moving along a cylindrical surface acquires an angular velocity ω directed radially when it collides with the wall.

According to Eq. (3), the transverse force does not acquire a radial component. The assumption of the constant v transverse to the flow is based on a rather large value of the corresponding Reynolds number, under which conditions the relative thickness of the x boundary layers is small. Projecting Eq. (4) onto the coordinate axes, we find

$$\frac{du_x}{d\tau} = u_y - \lambda u_x; \qquad \frac{du_y}{d\tau} = -u_x - \lambda u_y,$$
$$\tau = \frac{t}{\varkappa}, \qquad \frac{1}{\varkappa} = 2\frac{\rho}{\rho_1}\omega, \qquad \lambda = \frac{3}{8}\frac{\xi}{d}\frac{|u|}{\omega}. \tag{5}$$

(For convenience here, the sign of ω has been changed, so that ω is assumed constant when the particle rolls from left to right along the horizontal plane.) The solution of system (5) is

$$W_x = v + e^{-\lambda\tau}(A\cos\tau + B\sin\tau), \quad W_y = e^{-\lambda\tau}(B\cos\tau - A\sin\tau). \tag{6}$$

where A and B are arbitrary constants.

To determine the effect of the Magnus force, we consider the following example. We assume that there is no resistance force, so that $\lambda = 0$. We further assume that $w_x = w_y = x = y = 0$ when $\tau = 0$, and that the second plane is absent. In this case, we have $B = 0$, $A = v$, and

$$w_x = v(1 - \cos\tau), \qquad w_y = v\sin\tau,$$
$$x = \varkappa v(\tau - \sin\tau), \qquad y = \varkappa v(1 - \cos\tau).$$

The last two equations are the parametric equations of a cycloid. Under the influence of the Magnus force, the particle starts to move vertically upward, and then turns to the right. At $\tau = \pi$, it reaches its highest point $x = 2\varkappa v$, at which $w_x = 2V$, and then descends along a trajectory symmetric with respect to the ascent trajectory. During each cycloid spacing, the Magnus force changes sign twice: at $\tau = \pi/2$ and $3\pi/2$, when $w_x = v$. In the gap $\pi/2 < \tau < 3\pi/2$, the particle moves more rapidly than the flow, so the Magnus force is directed vertically downward. The average horizontal particle velocity over the period is $\langle w_x \rangle = v$ despite the absence of drag. The collision of the particle with the plane may be interpreted as the result of an impulsive force with impulse I. We have the following relation:

$$m\Delta w_x = I_x, \qquad m\Delta w_y = I_y, \qquad J\Delta\omega = -\tfrac{1}{2}\,dI_x. \tag{7}$$

Here we have $\Delta w = w^+ - w^-$, where the plus sign and minus sign denote quantities before and after the collision, respectively; and $J = 0.1\,md^2$ is the moment of inertia of the sphere. It follows from the first and last of Eqs. (7) that

$$\Delta w_x = -\tfrac{1}{5}\,d\Delta\omega. \tag{8}$$

System (7) contains five unknowns: w_x^+, w_y^+, ω^+, I_x, and I_y; we must therefore find two more relations.

For the first relation, we use the Newtonian equation

$$w_y^+ = -\mu w_y^- , \tag{9}$$

where μ is the Newtonian "coefficient of restitution."

To find the missing relation, we note that during a frictional collision, two cases are possible: first, the particle stops sliding along the surface during the collision, and

$$w_x^+ = {}^1/_2\, \omega^+ d ; \tag{10}$$

second, the particle continues to slide after the collison. In the latter case, the "dry-friction" law is used: $I_x = fI_y$, where f is the coefficient of friction. Which case actually occurs depends on the surface properties and on the angle at which the particle strikes the plane. Below, we consider only the first possibility, and use Eq. (10).

The problem of seeking the steady-state, i. e., periodic, motion of the particle in a flow between the two planes is formulated in the following manner. We consider the particle motion between collisions. At the instant the particle leaves the lower plane, we assume $\tau = 0$, $w_x = \overset{\circ}{w}_x$, $w_y = \overset{\circ}{w}_y$. The indices will be omitted from the corresponding quantities for the instant at which the particle collides with the upper surface.

We assume the angular velocity ω of the particle is constant during the motion. The periodicity condition requires the following:

$$w_x{}^\circ = w_x{}^+, \quad w_y{}^\circ = w_y{}^+, \quad w_x = w_x{}^-, \quad w_y = -w_y{}^-, \quad \Delta\omega = 2\omega .$$

Applying these equations to Eqs. (8)–(10), we find

$$w_x{}^\circ = {}^5/_9\, w_x, \quad w_y{}^\circ = \mu w_y, \quad {}^1/_2\, \omega d = w_x{}^\circ . \tag{11}$$

Equations (11) show how the particle velocity should increase during its free flight under the influence of the Magnus force in order that this increase compensate for the loss during the collision and in order that periodic motion be possible.

From conditions (11), we can determine the constants A and B in (6):

$$A = -\frac{4v}{\Delta}(1 - \mu e^{-\lambda\tau}\cos\tau), \qquad B = \frac{4v}{\Delta}\mu e^{-\lambda\tau}\sin\tau ,$$
$$\Delta = 9 + 5\mu e^{-2\lambda\tau} - (9 + 5\mu)\, e^{-\lambda\tau}\cos\tau . \tag{12}$$

To determine τ we specify the distance between planes. Integrating the second of relations (6), and using (12), we find

$$h = \frac{4\varkappa v}{(1+\lambda^2)\,\Delta}\,[1 + \mu e^{-2\lambda\tau} - (1+\mu)\, e^{-\lambda\tau}\cos\tau - \lambda\,(1-\mu)\, e^{-\lambda\tau}\sin\tau] . \tag{13}$$

The last of relations (11) yields

$$\frac{\omega d}{2} = \frac{5v}{\Delta}\,[1 + \mu e^{-2\lambda\tau} - (1+\mu)\, e^{-\lambda\tau}\cos\tau] . \tag{14}$$

The unknowns ω and τ should be determined from Eqs. (13) and (14).

We can calculate the average translational velocity of the particle, $\langle w_x \rangle = x/\varkappa\tau$. Integrating the first of relations (6), we find

$$\frac{(1+\lambda^2)\,\langle w_x \rangle}{v} = 1 + \lambda^2 - \frac{4}{\tau\Delta}\{\lambda\,[1 + \mu e^{-2\lambda\tau} - (1+\mu)\, e^{-\lambda\tau}\cos\tau] + (1-\mu)\, e^{-\lambda\tau}\sin\tau\} .$$

Using Eqs. (13) and (14), we convert this to

$$\langle w_x \rangle = v - \frac{2}{5}\frac{\omega d}{\lambda\tau}(1-\alpha) \qquad \left(\alpha = 5\,\frac{\rho}{\rho_1}\,\frac{h}{d}\right). \tag{15}$$

Using the notation

$$\beta = 3/80\,\zeta\,|u|\,v\,, \tag{16}$$

we find

$$\omega = 0\;\beta v/\lambda d\,. \tag{17}$$

Then Eqs. (13) and (14) become

$$1 - \frac{\lambda(1-\mu)\,e^{-\lambda\tau}\sin\tau}{1+\mu e^{-2\lambda\tau}-(1+\mu)\,e^{-\lambda\tau}\cos\tau} = d\,(1+\lambda^2)\,,$$
$$\lambda\;\frac{1+\mu e^{-2\lambda\tau}-(1+\mu)\,e^{-\lambda\tau}\cos\tau}{9+5\mu e^{-2\lambda\tau}-(9\mu+5)\,e^{-\lambda\tau}\cos\tau} = \beta\,. \tag{18}$$

Here λ and τ are unknowns. Using Eqs. (12), (13), and (14), we find

$$B = w_y{}^{\circ} = \frac{4\mu}{1-\mu}\left[\frac{\omega d}{10\lambda} - \frac{(1+\lambda^2)\,h}{4\varkappa\lambda}\right] = \frac{2}{5}\,\frac{\mu}{1-\mu}\,\frac{\omega d}{\lambda}\,[1-\alpha\,(1+\lambda^2)].$$

In order to have $w_y^{\circ} \geq 0$, we must have $\alpha \leq 1/(1+\lambda^2)$. When $w_y^{\circ} = w_y = 0$, we have $\tau = \pi$ (the maximum possible particle velocity is $\langle w_x\rangle = \nu(1 - 4\pi^{-1}\alpha\beta)$. When $\alpha > 1/(1+\lambda^2)$, there can be no periodic motion.

It is difficult to find the exact solution of (18), so we limit ourselves to an approximate solution; i. e., we set

$$\langle w_x\rangle \approx \frac{w_x{}^{\circ} + w_x}{2}\,,$$
$$\langle w_y\rangle = \frac{h}{\varkappa\tau} \approx \frac{w_y{}^{\circ} + w_y}{2}\,. \tag{19}$$

The use of Eqs. (19) does not produce a large error for α values which are not too large, since the functions $x(\tau)$ and $y(\tau)$ increase monotonically for $\alpha < 1/(1+\lambda^2)$. Substituting (11) into (19), we find

$$\langle w_x\rangle = \frac{7}{5}\,w_x{}^{\circ}, \quad \langle w_y\rangle = \frac{h}{\varkappa\tau} = \frac{1+\mu}{2\mu}\,w_y{}^{\circ},$$
$$\omega d = \frac{10}{7}\,\langle w_x\rangle,$$
$$\frac{h}{\varkappa\tau} = \frac{1}{5}\,\frac{1+\mu}{1-\mu}\,\frac{\omega d}{\lambda}\,[1-\alpha\,(1+\lambda^2)]\,.$$

Furthermore, using (17), we find

$$\lambda = 7\beta v/\langle w_x\rangle\,.$$

Substituting the ω, λ, and τ values into Eq. (15), we find the cubic equation

$$(1-\alpha)^2 z^3 + \alpha\,[b - a\,(1-\alpha)]\,z - \alpha b = 0\,,$$
$$z = \frac{\langle w_x\rangle}{v}\,, \quad a = 49\beta^2, \quad b = \frac{7\,(1-\mu)}{2\,(1+\mu)}\,49\,\beta^2\,. \tag{20}$$

Equation (20) is conveniently analyzed through a consideration of the $\alpha(z)$ dependence. Some strightforward manipulations yield

$$\alpha = \frac{2z^3 + [b + (a-b)\,z] - \sqrt{[b + (a-b)\,z]^2 + 4b\,(1-z)\,z^3}}{2\,(z^3 + az)}\,, \tag{21}$$

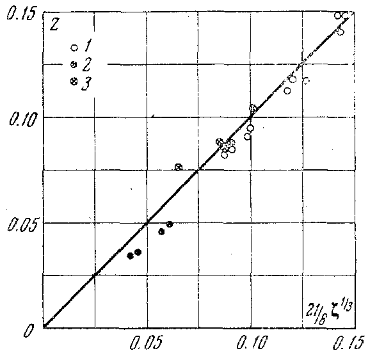
from which it follows that $\alpha(0) = 0$ and $\alpha(1) = 1/(1+a)$. When $\alpha \ll 1$, we find from (20) that

$$Z = (\alpha b)^{1/3} = {}^{21}/_{8}\,(\zeta)^{1/3}\left(\zeta = \frac{\xi^2}{15}\,\frac{\rho}{\rho_1}\,\frac{1-\mu}{1+\mu}\,\frac{h}{d}\,\frac{u^2}{v^2}\right). \tag{22}$$

This expression loses its validity at μ values close to unity. When $\mu = 1$, we have

$$Z = \left(\frac{a\alpha}{1-\alpha}\right)^{1/2} = \frac{21}{80}\,\xi\,\frac{|u|}{v}\left(\frac{\alpha}{1-\alpha}\right)^{1/2}.$$

We note that Eq. (22) can be found directly from Eqs. (6) and (11) by setting $\sin \tau \approx \tau$, $\cos \tau = 1$, and $e^{-\lambda\tau} \approx 1 - \lambda\tau$, which is a valid procedure when $\tau \ll 1$. The accompanying figure compares some experimental data and some data calculated from Eq. (23). The experiments were carried out in a vortex chamber about 300 mm in diameter.



Points 1 in the figure corresponds to plexiglas spheres 1 and 2 mm in diameter; points 2 corresponds to a steel sphere 1 mm in diameter ($\mu = 0.9$ and $\alpha = 0.01$); and points 3 correspond to a plexiglas sphere 3 mm in diameter ($\mu = 0.57$ and $\alpha = 0.02$). The agreement between theory and experiment is seen to be completely satisfactory. The experiments will be described in a separate communication.

REFERENCES

1. M. A. Gol'dshtik, "Secondary flows which arise near a sphere during its rotation in a viscous liquid," Inzh.-fiz. zh., no. 3, 1960.

2. M. A. Gol'dshtik, A. K. Leont'ev, and I. I. Paleev, "Experimental study of the motion of solid particles in a vortex chamber," Nauchn.-tekhn. inform. byulleten Leningr. Politekhn. in-ta, no. 2, 1960.

27 November 1967

Novosibirsk